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Plasma Velocity Determination by Electrostatic Probes

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Nomenclature

a = sheath radius
a' = local speed of sound
A = current collection surface area of probe
c_m = most probable thermal speed of the particles
e = basic unit of electrical charge
I = current collected by a probe which has a general orientation with respect to the plasma velocity
$I_\perp$ = ion current collected by a probe perpendicular to the plasma velocity
$I_\parallel$ = ion current collected by a probe parallel to the plasma velocity
J_n = Bessel function
k = Boltzmann's constant
m = mass of particle
N = particle number density
r = probe radius
T = temperature
$\mathbf{v}$ = velocity of flowing plasma
V_0 = probe potential relative to plasma potential
z = 1 + degree of ionization
Γ = gamma function
$\tilde{\gamma}$ = incomplete gamma function
γ = ratio of specific heats
γ_0^2 = $r^2/(a^2 - r^2)$
κ = $(v/c_m)\sin\theta$
θ = orientation angle of $\mathbf{v}$ with the longitudinal axis of the probe

Subscripts

e = electron species
i = ion species

Introduction

In a flowing plasma the ion current collected by a cylindrical electrostatic probe is a function of the angle between the longitudinal axis of the probe and the velocity vector of the plasma.[1-3] In the present Note this velocity dependence of ion current collection by an electrostatic probe is utilized to develop a method of determining the velocity of a flowing plasma from the Langmuir curves of two mutually perpendicular probes. This method allows one to determine the local plasma velocity simultaneously with the local measurements of the electron and ion density and the electron temperature.

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Mathematical Development

The current collected by a moving cylindrical electrostatic probe immersed in a stationary plasma has been determined by Kanal to be given by the expression[3]

$$I = \left[\frac{kT}{2\pi m}\right]^{1/2} NeA \frac{2}{\pi^{1/2}} e^{-\kappa^2} \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \left\{ e^{V_0} V_0^{-n/2} \times \Gamma\left(n + \frac{3}{2}, V_0(1+\gamma_0^2)\right) J_n(2\kappa V_0^{1/2}) + \frac{a}{r}\frac{\kappa^n}{n!} \tilde{\gamma} \times \left(n + \frac{3}{2}, \gamma_0^2 V_0\right)\right\}$$

where

$$\tilde{\gamma}(\nu,\chi) = \int_0^\chi e^{-t} t^{\nu-1} dt \tag{2}$$

$$\Gamma(\nu,\chi) = \int_\chi^\infty e^{-t} t^{\nu-1} dt \tag{3}$$

The assumptions of a cylindrical sheath of radius a and a Maxwellian velocity distribution referred to the moving coordinate system were employed. This expression may also represent the current collected by a stationary probe in a flowing plasma. If one assumes negligible sheath thickness, i.e., $a/r \to 1$, then $\gamma_0 \to \infty$, and Eq. (1) may be expressed as

$$I = \left[\frac{kT}{2\pi m}\right]^{1/2} NeA \frac{2}{\pi^{1/2}} e^{-K^2} \sum_{n=0}^{\infty} \left[\frac{\kappa^n}{n!}\right]^2 \Gamma\left(n + \frac{3}{2}\right) \tag{4}$$

Two orientations of a cylindrical electrostatic probe relative to the flowing plasma will be considered: 1) the longitudinal axis of the probe perpendicular to the plasma velocity vector, and 2) the longitudinal axis of the probe parallel to the plasma velocity vector. The current collected by the probe in orientations 1 and 2 is denoted $I_\perp$ and $I_\parallel$, respectively. For case 1, $\kappa = v/c_m$ whereas for case 2, $\kappa = 0$. Therefore,

$$I_\perp = \left[\frac{kT}{2\pi m}\right]^{1/2} NeA \frac{2}{\pi^{1/2}} e^{-(v/c_m)^2} \times \sum_{n=0}^{\infty} \left[\frac{(v/c_m)^n}{n!}\right]^2 \Gamma\left(n + \frac{3}{2}\right) \tag{5}$$

$$I_\parallel = \left[\frac{kT}{2\pi m}\right]^{1/2} NeA \tag{6}$$

For equal current collection surface area for the two probes, the ratio of current collected by a probe perpendicular to the plasma velocity to that collected by a probe parallel to the plasma velocity is thus given by

$$\frac{I_\perp}{I_\parallel} = \frac{2}{\pi^{1/2}} e^{-(v/c_m)^2} \sum_{n=0}^{\infty} \left[\frac{(v/c_m)^n}{n!}\right]^2 \Gamma\left(n + \frac{3}{2}\right) \tag{7}$$

Using the ratio test, it can be shown that the foregoing in-

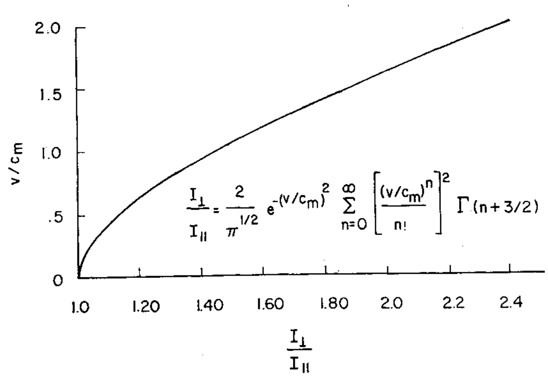

Fig. 1 Plot of $I_\perp/I_\parallel$ vs v/c_m.

finite series is convergent and thus can be truncated. Figure 1 is a plot of $I_\perp/I_{\parallel}$ vs v/c_m, where the first 50 terms of the series were retained.

We wish to utilize Eq. (7) to represent only the ion current collected by the two probes; c_m is then the most probable thermal speed of the ions, $(2kT_i/m_i)^{1/2}$. A condition upon which Eq. (7) is based is that the plasma sheath thickness is negligible. Since the thickness of the probe's plasma sheath decreases as the probe potential approaches the plasma potential, the experimentally determined ion saturation current region of the Langmuir curves determined by the two probes should be extrapolated to a probe potential point in the immediate neighborhood of the plasma potential. The extrapolated values of $I_\perp$ and $I_{\parallel}$ are used to determine v/c_m from Fig. 1. Knowing the ion temperature of the plasma, which is approximately equal to the static temperature of the gas, the plasma velocity can be determined.

Considering the ion temperature to be equal to the gas temperature, the mean thermal speed of the ions is approximately equal to the local speed of sound, i.e., $c_m = (2/\gamma z)^{1/2}a'$. From the measured ion current ratio $I_\perp/I_{\parallel}$, one obtains the Mach number based on the ion thermal speed v/c_m, which is approximately equal to the Mach number based on the speed of sound v/a'. Thus, Eq. (7) may be utilized to closely estimate the Mach number of the flowing plasma from Langmuir probe measurements.

Experimental Measurements

Langmuir probe measurements were performed in a supersonic argon plasma stream to determine the feasibility of the described method. The Mach number of the flowing plasma determined by the Langmuir probe technique was compared with the Mach number determined by an independent method, thereby eliminating the error which would be induced by static temperature measurements required for velocity comparison. The probes were constructed from 10-mil, electropolished surface finish, tungsten wire, and were each $\frac{1}{8}$ in. in length. The tests were performed at volume flow rates ranging from 5 to 30 ft^3/hr with corresponding total pressures of 0.32 to 3.68 torr. The total temperature was estimated from alumel-chromel thermocouple measurements to range from approximately 1000°R at the lowest flow rate to 2000°R at the highest flow rate. The gas density ranged from 1.4×10^{-5} to 6.7×10^{-5} lbm/ft^3. The electron number density determined by Langmuir probe measurements ranged from 1.6×10^{13} cm^{-3} at the lowest flow rate to 9.6×10^{13} at the highest flow rate. Figure 2 illustrates the manner in which the ion current collected by a Langmuir probe with its longitudinal axis parallel to the flow varied with freestream electron number density. Since N_e was increased by increasing the volume flow rate, which increased the gas velocity, Fig. 2 indicates that $I_{\parallel}$ is not velocity dependent.

The potential of the Langmuir probes was referenced to the anode potential of the plasma generator. Experimentally it was found that the plasma potential differed from the anode potential of the plasma generator by less than 1 v. Thus, the experimentally determined ion saturation current region of the Langmuir curves determined by the two probes was extrapolated to the anode potential rather than to the plasma potential. This procedure reduced the problem of probe heating since only the ion saturation region of the probe characteristic was now required. The geometric areas of the two probes were the same; however, the ion current collected by the back half of the perpendicular probe will be substantially less than the ion current collected by the front half.[4] It was thus assumed that only the front half of the perpendicular probe collected ion current.

Figures 3a–3c present experimentally determined Langmuir characteristics of $I_\perp$ and $I_{\parallel}$ for three different flow rates. The experimentally determined $I_\perp$ has been increased by a factor of 2 to provide for equivalent current collection surface areas for the two probes. Extrapolation to the anode potential is illustrated. The Mach number of the flowing plasma for the three flow rates determined by the Langmuir probe technique was 1.12, 1.18, and 1.18. For the same flow rates, a Mach number of 1.20, 1.23, and 1.25, respectively, was de-

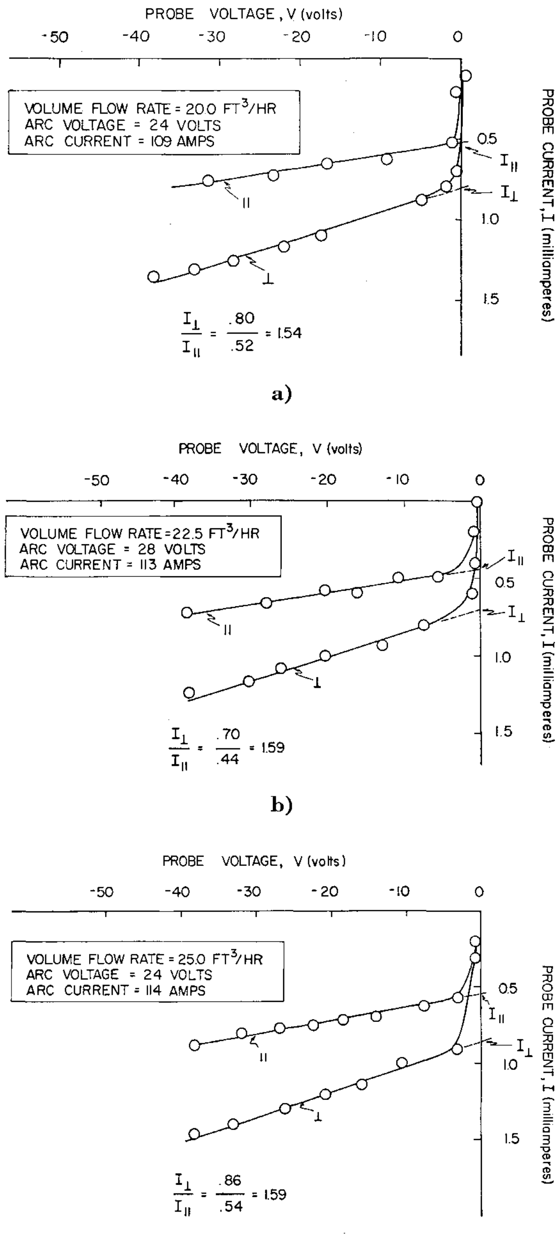

Fig. 3 Probe current vs probe voltage, ion saturation region.

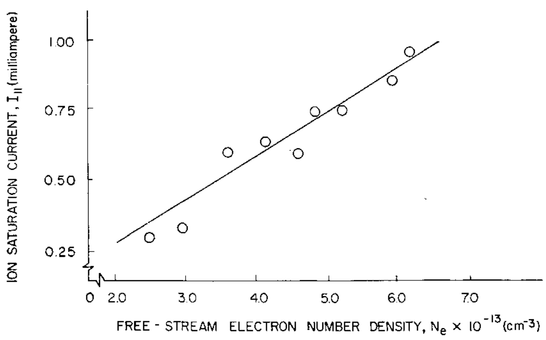

Fig. 2 Variation of $I_{\parallel}$ with the freestream N_e.

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Analysis of Turbulence by Schlieren Photography

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In this Note we consider the relationship of the statistical parameters that describe the turbulent density variations to the intensity variations on the schlieren photograph of a turbulent flow. We obtain the solution using ray optics. We set the schlieren system with the knife edge oriented parallel to the z axis, and light rays starting in the x direction. The medium through which the rays pass is characterized by an index of refraction, $n = \bar{n} + \mu(x,y,z)$, where μ is a small stochastic quantity describing the variation of the index of refraction from its mean value. Using the equation of the eikonal, the phase variation to the first order of μ is

$$\delta\phi = \int_0^L \mu(x,y,z)dx$$

The integral is to be taken along the straight-line path, $0 \leq x \leq L$, traversed by the rays in the absence of the stochastic variation of the index of refraction. The local deviation of the beam from this straight path (to the first order of μ) is, therefore,

$$\delta\theta = \frac{\delta\phi}{\partial y} = \int_0^L \frac{\partial\mu(x,y,z)}{\partial y}\, dx$$

This expression is the component of angular variation in the y direction (perpendicular to the knife edge). The variation of intensity, $I_1 = \bar{I} - I(y_1,z_1)$, at a point on the image is proportional to $(\delta\theta)$. Thus,[1]

$$I_1 \propto \int_0^L \frac{\partial\mu(x_1,y_1,z_1)}{\partial y_1}\, dx_1$$

The normalized autocorrelation function for the intensities on the photographic plate is now defined as

$$R_{12} = \frac{\langle I_1 I_2\rangle}{\langle I^2\rangle} = \int_0^L \int_0^L \left\langle \frac{\partial\mu(x_1,y_1,z_1)}{\partial y_1}\, \frac{\partial\mu(x_2,y_2,z_2)}{\partial y_2} \right\rangle \times dx_1 dx_2 \Bigg/ \left\langle \left\{ \int_0^L \frac{\partial\mu(x_1,y_1,z_1)}{\partial y_1}\, dx_1 \right\}^2 \right\rangle$$

Proceeding in a standard way,[2] this result may be written in a more tractable form by introducing the relative coordinates $x = x_1 - x_2$, $y = y_1 - y_2$, $z = z_1 - z_2$ and the definition for the correlation function of the refractive index variations

$$C(x,y,z) = C(x_1 - x_2, y_1 - y_2, z_1 - z_2) = \langle \mu(x_1,y_1,z_1)\mu(x_2,y_2,z_2)\rangle \ \langle\mu^2\rangle$$

where we assume isotropic and homogenous turbulence. Thus,

$$R_{12} = \int_0^L \int_0^L \left[\frac{\partial^2}{\partial y^2} C(x,y,z)\right]_{P_n} dx_1 dx_2 \Bigg/ \int_0^L \int_0^L \left[\frac{\partial^2}{\partial y^2} C(x,y,z)\right]_{P_n} dx_1\, dx_2$$

where $P_n = (y_1 - y_2)\hat{y} + (z_1 - z_2)\hat{z}$ is the point at which the integrand is to be evaluated. Further simplification can be made by introducing the integral relation

$$\int_0^{L_1} dx_1 \int_0^{L_2} f(x)\, dx_2 = \int_0^{L_1} (L_1 - x)\, f(x)dx + \int_0^{L_2} (L_2 - x)f(x)dx - \int_0^{L_1 - L_2} (L_1 + L_2 - x)f(x)dx$$

If $L_1 = L_2 \gg l$, the scale of turbulence, the upper limit may be replaced with ∞. Thus,

$$\int_0^L \int_0^L f(x)dx_1dx_2 \doteq 2L \int_0^\infty f(x)dx$$

The final result for the autocorrelation of the intensities is

$$R_{12} = \int_0^\infty \left(\frac{\partial^2}{\partial y_2} C(x,y,z)\right)_{P_n} dx \Bigg/ \int_0^\infty \left(\frac{\partial^2}{\partial y^2} C(x,y,z)\right)_{P_n = 0} dx$$

In arriving at this final equation, it has been assumed that the path length L traversed by the light rays is large compared to the range of appreciable $C(x,y,z)$.

In order to understand the significance of the result, let the correlation function for the refractive index variations have the Gaussian form

$$C(r) = \exp(-r^2/l^2)$$

Writing the autocorrelation function, it is found that the integrals separate, yielding

$$R_{12} = \frac{\partial^2}{\partial y^2} \exp[-(y^2 + z^2)/l^2]|_{P_n} \Bigg/ \frac{\partial^2}{\partial y^2} \exp[-(y^2 + z^2)/l^2]|_{P_n = 0}$$

$$= (1 - 2y^2/l^2) \exp[-(y^2 + z^2)/l^2]$$

If positions perpendicular to the knife edge are taken, $z = 0$, $P_n = y$; and the result is

$$R_{12}{}^{\perp} = (1 - 2y^2/l^2) \exp(-y^2/l^2)$$

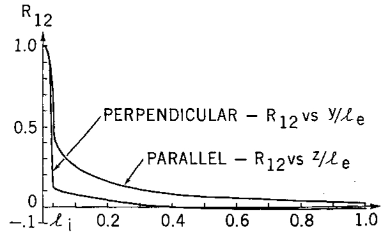

Fig. 1 Autocorrelation for positions parallel and perpendicular to the knife edge.

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